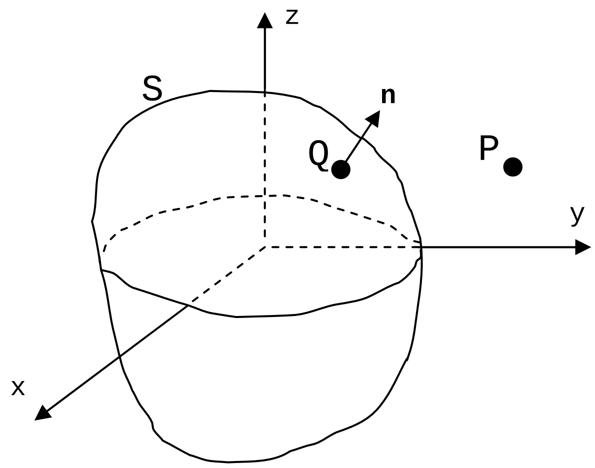
AXISYMMETRICAL OPENBEM

Theoretical background



- The model makes use of the Helmholtz integral equation:

$$C(P)p(P) = \int_S \left(p(Q)\frac{\partial G(R)}{\partial n} + ikz_0 v(Q)G(R) \right) dS + 4\pi p^I(P)$$

Relates pressure/velocity (*p, v*) on the surface (*S*) of a body (*Q*) and an incoming wave (p^I) with the pressure on external point *P*. *k* is the wavenumber, $G(R) = e^{-ikR} / R$ the Green function, *R* is $|P - Q|$, z_o the characteristic impedance (ρc) and *C(P)* the solid angle seen from *P* (4π for *P* outside *S*, *0* inside and 2π on *S*, if *S* is smooth).

Both harmonic time variation and infinite homogeneous medium are supposed (p satisfies $\nabla^2 p + k^2 p = 0$).

Axisymmetrical formulation with non-axisymmetrical boundary conditions

The Helmholtz integral equation is expressed in cylindrical coordinates and the dependence on the generator is separated from the dependence on rotational coordinate. In other words, if the body/bodies are symmetrical around a common axis, the surface integral can be reduced to a line integral along the generator.

The use of a cosine expansion of *p* and *v* in orthogonal terms, allows the isolation of the singularities contained in the revolution integrals.

$$p = \sum_{m=0}^{\infty} p_m \cos m\theta$$

$$v = \sum_{m=0}^{\infty} v_m \cos m\theta$$

In this way the rotational dependence can be solved analytically, so that only the generator has to be discretized. The calculation produces the coefficients p_m and v_m. The final approximate solution is obtained by summing up a sufficient number of terms.

If the integral equation above is discretized into elements along the body(s) generator, it can be converted into a system of equations. This system can be expressed as a matrix equation:

$$C\ p_m = A_m\ p_m + i\ k\ z_o\ B_m\ v_m + 4\ \pi\ p_m^{\ I}\ \ ;\ \ m = 0,1,2,...$$

where m represents the term of the cosine expansion. The pressure and normal velocity on the nodes along the generator are p_m and v_m, p^I_m is the pressure of the incident wave on the nodes in the absence of the body. The matrices A_m and B_m are only function of the geometry, frequency and m term; this makes possible to reuse its calculation for different boundary conditions.

This equation could be expressed by replacing the pressures p_m by velocity potential. To adjust, the ikz_o factor of the velocity term should be removed. However, this is not very relevant for the purpose at B&K, since the calculations are usually made for sound pressure.

The C constants are represented by a diagonal matrix. This term in the left-hand side can be subtracted in the right-hand side in this way:

$$0 = A'_m\ p_m + i\ k\ z_o\ B_m\ v_m + 4\ \pi\ p_m^{\ I}\ \ ;\ \ m = 0,1,2,...$$
$$where\quad A'_m = A_m - C$$

This is done automatically in the 'BemEquat' function, therefore its output is A'_m. In other words, they are only useful for the code developers. We will call A'_m just A_m from now on, but the difference should be clear.

Solving the system in Matlab

This matrix form can be written and solved directly in Matlab. The equation must be composed with the known data and solved to obtain the unknown data. The available boundary conditions and excitations can vary from problem to problem. For example, a scattering problem with a rigid body (no normal surface velocity, infinite impedance) with an external excitation (sound sources) would not have velocity term, therefore it can be written:

$$0 = A_m\ p_m + 4\ \pi\ p_m^{\ I}$$
$$\Rightarrow p_m = A_m^{-1}\left(-4\ \pi\ p_m^{\ I}\right);\ \ m = 0,1,2,...$$

On the other hand, a radiation problem with no external excitation would have normal velocities defined on the boundary. It would be:

$$0 = A_m\, p_m + i\, k\; z_o\, B_m\, v_m$$
$$\Rightarrow p_m = A_m^{-1}\left(-i\, k\; z_o\, B_m\, v_m\right)\ ;\quad m = 0,1,2,...$$

Velocity and impedance

The velocity and impedance boundary conditions must be implemented through the velocity term:

$$v_m = Y\, p_m + v_m^o\ ;\quad m = 0, 1, 2, ...$$

$$\begin{pmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mM} \end{pmatrix} = \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & Y_M \end{pmatrix} \begin{pmatrix} p_{m1} \\ p_{m2} \\ \vdots \\ p_{mM} \end{pmatrix} + \begin{pmatrix} v_{m1}^o \\ v_{m2}^o \\ \vdots \\ v_{mM}^o \end{pmatrix}$$

The values are expanded here for all the nodes (M). The Y_i are the admittances on the nodes (impedances^{-1}), and the v^o_{mi} are the fixed normal velocities on the nodes in radiation problems. Only one of the terms in the right-hand side will be usually present, but in some practical cases both could exist. This happens when lining material covers the surface, and therefore the normal surface velocity is different from the air normal velocity.

As an example, let us consider a scattering case with finite surface impedance:

$$0 = A_m\, p_m + i\, k\; z_o\, B_m\, v_m + 4\,\pi\; p_m^{\;I}\ ;\quad m = 0,1,2,...$$
$$\Rightarrow 0 = A_m\, p_m + i\, k\; z_o\, B_m Y\; p_m\; + 4\,\pi\; p_m^{\;I}$$
$$\Rightarrow p_m = \left(A_m + i\, k\; z_o\, B_m Y\right)^{-1}\left(-4\,\pi\; p_m^{\;I}\right)$$

Solving for the velocity

Up to now, it is assumed we solve the pressure, but it may happen that the pressure is given at the boundary and we must calculate the velocity. A simple example is a pressure release boundary.

When pressure and velocity boundary conditions are mixed along the boundary, the system of equations must be rearranged in order to solve it. In this example one node has a pressure boundary condition:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & -b_{1i} & \cdots & a_{1M} \\ a_{21} & a_{22} & & -b_{2i} & & a_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & -b_{Mi} & \cdots & a_{MM} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ v_i \\ p_M \end{pmatrix} =$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & -a_{1i} & \cdots & b_{1M} \\ b_{21} & b_{22} & & -a_{2i} & & b_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{M1} & b_{M2} & \cdots & -a_{Mi} & \cdots & b_{MM} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ p_i \\ v_M \end{pmatrix} + \begin{pmatrix} p_1^I \\ p_2^I \\ \vdots \\ p_M^I \end{pmatrix}$$

The index *m* that indicates expansion term has been removed for simplicity.

Expansion in m-terms

As explained before, pressure and velocity are expanded in cosine series. In this way they can have a non-axisymmetrical behavior. This implies solving a different system of equations for every value of *m*, and assembling the solution using the coefficients of the expansion obtained.

This is how the different terms are handled:

-The incoming field from external sources is calculated for every *m* value by the function ‘incoming’. It must be called for every new *m* term.

-The normal velocity boundary condition must also be expanded. The function ‘expand’ uses FFT to obtain the m terms in this case. An example is the half-ring source.

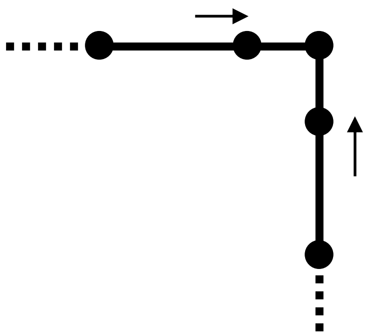
-The coefficient matrices *A* and *B* must also be obtained for every *m* value. The function ‘BEMequat’ is able to calculate them together for all m-values, saving computer time.

-The impedance cannot be expanded in cosine series. Therefore, only axisymmetrical impedances are allowed.

The number of m values needed depends on the accuracy needed in the rotational direction, the frequency and the smoothness of the rotational variation. A few tries should give an idea for a specific problem.

Quarter point technique

The acoustic variables near a square edge vary as $r^{2/3}$, where *r* is the distance to the edge. The quarter point technique consists in a displacement of the central node in the adjacent element towards the edge, up to a certain distance (0.275 times the element length) that makes the shape functions very similar to the actual function. This makes the solution more accurate with fewer elements.



The quarter point technique is implemented during the meshing of the generator in 'nodegen'.

Diaphragm modeling

An approximation that only takes into account the first mode of vibration can be used. The diaphragm acoustic impedance is considered infinite. This is called in Peter's program 'blocked diaphragm' calculation.

Both sensitivity and pressure response are affected by a parabolic weighting function $f(r) = 1- (r^2/a^2)$, where *r* is the distance to the center and *a* the radius of the diaphragm. If we call the radius of the backplate *b*, the microphone complex pressure 'output' is:

$$\bar{p} = \frac{\int_0^b p(r)\, f(r)\, r\, dr}{\int_0^b f(r)\, r\, dr}$$